The Geometrical Generating of Gravity

Cheng Jian-Feng [a], Huang Xin-bing[b]

[a] *Institute of Theoretical Physics, Chinese Academy of Sciences P.O.Box 2735, Beijing 100080, China*

[b] *Institute of High Energy Physics, Chinese Academy of Sciences P.O.Box 918, Beijing 100039, China*

Abstract

This article analyzes why the energy-momentum tensors are **not** the source of gravity and the dynamical variable of gravity is the affine connection instead of the metric. We derive new gravitational equations with the dimensions of three derivatives from the gravitational action, which has the same solution with the Einstein equations in the vacuum. We also discuss the connection between the new gravitational equations and the Einstein equations.

PACS number: 04.50.+h

1 Motivation and introduction

Mass played an important role in the development of gravity from Newton to Einstein. From the fact that the bodies fall at a rate independent of their mass, Newton concluded the force exerted by gravity is proportional to mass of the body on which it acts, which led to the first gravitational equation in the history:

$$F = G\frac{m_1 m_2}{r^2} ,$$

or

$$\nabla^2\phi = 4\pi G\rho , \tag{1}$$

where ρ is mass density, G is Newton's constant and ϕ is the Newtonian potential. Thus mass became the source of gravity naturally. Impressed with the observed equality of gravitational and inertial mass, Einstein found an elegant entrance to the curved space-time—the Principle of Equivalence. He developed Newton's idea and wrote down the energy-momentum tensors on the right of the equations as the source of gravity, but with which he was not satisfied because the energy-momentum tensors are not the geometrical variable. He got the second gravitational equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} . \tag{2}$$

However, Newton accepted mass as the source of gravity according to experience completely. Is gravity really exerted by mass, or more exactly, the energy-momentum tensors? Can the motion equation 's independence of the energy-momentum tensors demonstrate that the source of gravity is the energy-momentum tensors? Furthermore, there is one sharper problem. After the

foundation of the general theory of relativity, Einstein was annoyed with a problem: it is hard to give a good definition to the energy-momentum of the gravitational field. As the time went on, the problem did not get a final answer always and became seriously in the quantum theory of gravity. The gravitational field interacts with the matter field and itself because it interacts with anything that carries the energy-momentum tensors. Now that we can not describe the gravitational energy and momentum, we can not describe the self-interaction of gravity in the classical theory of gravity, we also can not describe the self-interaction in the quantum theory of gravity. Then what is the meaning of the nonlinear terms in the corresponding action, the Einstein-Hilbert action $-\frac{1}{16\pi G}\int d^4x\sqrt{-g}R(x)$? As an important result of this is that we can't calculate the quantum corrections of gravity from the quantum theory of the general theory of relativity directly, that is to say, we can not achieve a good quantum theory of gravity from quantizing the general theory of relativity directly.

In 1954 Yang and Mills brought forward the gauge field theory [1], which tell us that interaction is determined by symmetry as

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\alpha\mu\nu}F^{\alpha\mu\nu} - \mathcal{L}_M(\psi,\ D_\mu\psi)\ , \tag{3}$$

which is called the gauge principle. In the 1960's people became to realize the gravitational field is also some gauge field [2]. In 1974 Yang pointed out the gauge group of the gravitational field is group $GL(4,R)$, instead of the localized group $SO(3,1)$ [3]. The matter field in the Riemann manifold then becomes the vector representation of $GL(4,R)$, which makes itself have property of geometry and need a new index to describe. The gravitational field interacts with the matter field through the general covariance which turns the ordinary derivative into the covariant derivative. The geometrical property of the matter field is embodied in the gravity, or the curved space-time. Because of the existing of gravity, the bodies move in the curved space-time through the geodesic independent of their mass, which makes the force exerted by gravity proportional to their mass in the classical mechanics, according to which Newton accepted mass as the source of gravity. In the macroscopic scale mass or the energy-momentum tensors seem to be the source of the gravitational field and need a Newton's constant for dimension. As a result, we can not write the energy-momentum tensors on the right of the gravitational equations as the source of gravity and the general theory of relativity is just a good approximation in the macroscopic scale, which can not describe gravity well in high energy.

In Wu-Yang 's table [4] what corresponds to the gauge potential is the connection in the manifold, likewise, we conclude what corresponds to the gravitational potential is the affine connection in the Riemann manifold, instead of the metric and the dynamical variable of the gravitational field is the affine connection, which is not determined by the metric only for torsion. As a result, the affine connection $\Gamma^\lambda_{\mu\nu}$ corresponds to the gauge potential A^α_μ and the Riemann-Christoffel curvature tensor $R_{\lambda\mu\nu\rho}$ corresponds to the gauge-field tensor $F^\alpha_{\mu\nu}$ in the non-Abelian gauge field, therefore, the left of gravitational field equations should be the covariant derivative of the Riemann-Christoffel tensor $R_{\lambda\mu\nu\rho}$[3]. The new gravitational equations, with the dimension of three derivatives, can embody the self-interaction of gravity naturally in the non-linear terms of $D^\rho R_\rho{}^{\lambda\mu\nu}$, nor does the Einstein equations. Furthermore, it's easier to renormalize gravity because the coupling constant is of no dimension in the framework of new theory.

2 Other fields in the Riemann manifold

Gravity demonstrates that matter and space-time are indivisible. Because of the existing of gravity, all fields and wave functions "live" in the Riemann manifold, which need new Lorentz indices to describe them , then the spinor field ψ_α changes into ψ^μ_α; the ordinary gauge field $A_{\alpha\nu}$ changes into $A^\mu_{\alpha\nu}$; the scalar field ϕ_α changes into ϕ^μ_α. They are all vectors in the Riemann space-time.

The special theory of relativity is also valid in the locally inertial coordinate system.

3 Derivation of the gravitational field equations

We choose the spinor field as an example to discuss the matter field.

In the Riemann space-time, the proper time is

$$d\tau^2 \equiv -g_{\mu\nu}dx^\mu dx^\nu$$

where $g_{\mu\nu}$ is the metric tensor.

We define

$$g \equiv \text{Det } g_{\mu\nu} \ , \tag{4}$$

so the invariant volume is $\sqrt{-g}dx^4$.

In order to make a vector A^i has covariant derivative, we define

$$D_\mu A^i \equiv \partial_\mu A^i + \Gamma^i{}_{\mu j} A^j \ , \tag{5}$$

then we can calculate the transformation of Γ as

$$\Gamma^{*i}_{jk} = \frac{\partial x^{*i}}{\partial x^l}\frac{\partial x^m}{\partial x^{*j}}\frac{\partial x^n}{\partial x^{*k}}\Gamma^l_{mn} - \frac{\partial^2 x^{*i}}{\partial x^s \partial x^t}\frac{\partial x^s}{\partial x^{*j}}\frac{\partial x^t}{\partial x^{*k}} \ , \tag{6}$$

which contains two parts, symmetric term $\overline{\Gamma}^i_{jk}$ and asymmetric term $\hat{\Gamma}^i_{jk}$(between j and k). The symmetric term $\overline{\Gamma}^i_{jk}$ is determined by the metric tensor $g_{\mu\nu}$ as

$$\overline{\Gamma}^i_{jk} \equiv \frac{1}{2}g^{im}[\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}] \ . \tag{7}$$

Γ^i_{jk} is not a tensor but can construct a tensor $R^\lambda{}_{\mu\nu\rho}$ as

$$R^\lambda{}_{\rho\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\eta}\Gamma^\eta_{\nu\rho} - \Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\rho} \ , \tag{8}$$

usually called the Riemann-Christoffel curvature tensor, We use the affine connection $\Gamma^\lambda_{\mu\nu}$ to construct *covariant derivatives* of high tensors $T^{\rho\sigma\cdots}{}_{\kappa\lambda\cdots}$:

$$D_\nu T^{\rho\cdots}{}_{\kappa\cdots} \equiv \partial_\nu T^{\rho\cdots}{}_{\kappa\cdots} + \Gamma^\rho_{\nu\lambda} T^{\lambda\cdots}{}_{\kappa\cdots} + \cdots - \Gamma^\mu_{\nu\kappa} T^{\rho\cdots}{}_{\mu\cdots} - \cdots \ . \tag{9}$$

The general covariance induces that all vectors and tensors become representation of the general linear group GL(4,R), which include the matter field ψ^μ. When the coordinates have

a infinitesimal transformation $x^{*\mu} = x^\mu + \epsilon^\mu$, they have the following local transformation accordingly as

$$\delta\Gamma^\lambda_{\mu\nu} = -\partial_\mu\partial_\nu\epsilon^\lambda + \partial_\rho\epsilon^\lambda\Gamma^\rho_{\mu\nu} - \partial_\mu\epsilon^\rho\Gamma^\lambda_{\rho\nu} - \partial_\nu\epsilon^\rho\Gamma^\lambda_{\mu\rho} \ , \tag{10}$$

$$\delta\psi^\mu_\alpha = \partial_\rho\epsilon^\mu \ \psi^\rho_\alpha \tag{11}$$

where α is the spinor index. From (10) we can see that the affine connection contains the adjoint representation of $GL(4,R)$. The covariant derivative transforms as

$$\delta D_\nu\psi^\mu_\alpha = -\partial_\nu\epsilon^\rho D_\rho\psi^\mu_\alpha + \partial_\rho\epsilon^\mu D_\nu\psi^\rho_\alpha \ . \tag{12}$$

Analogous with (3), we write down the total action of the gravitational field and the matter field as

$$\begin{aligned} I &= \int d^4x\sqrt{-g} * (\mathcal{L}_G + \mathcal{L}_M) \\ &= -\int d^4x\sqrt{-g} * [\frac{1}{4}\ R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} + \mathcal{L}_M] \end{aligned} \tag{13}$$

where

$$\begin{aligned} \mathcal{L}_G &= -\frac{1}{4}R_{\lambda\rho\mu\nu}R^{\lambda\rho\mu\nu} \\ &= -\frac{1}{4}(\partial_\mu\Gamma^\lambda_{\nu\rho} - \partial_\nu\Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\eta}\Gamma^\eta_{\nu\rho} - \Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\rho})^2 \ , \end{aligned} \tag{14}$$

$$\begin{aligned} \mathcal{L}_M &= -g_{\rho\lambda}\bar{\psi}^\rho(m + \gamma^\mu D_\mu)\psi^\lambda \\ &= -g_{\rho\lambda}\bar{\psi}^\rho(m\delta^\lambda_\nu + \gamma^\mu[\delta^\lambda_\nu\partial_\mu + \Gamma^\lambda_{\mu\nu}])\psi^\nu \ . \end{aligned} \tag{15}$$

(m is mass of the spinor particle, $\mu\nu\omega$ are the Lorentz indices and the spinor indices are omitted). The nonlinear terms in $\mathcal{L}_G$ represent the self-interaction of gravity.

We use the variational principle to (13), keeping ψ^μ fixed and $\Gamma^\lambda_{\mu\nu}$ varied, we get

$$\begin{aligned} \delta I &= -\delta\int d^4x\sqrt{-g} * [\frac{1}{4}\ R_{\lambda\rho\mu\nu}R^{\lambda\rho\mu\nu} + g_{\rho\lambda}\bar{\psi}^\rho(m\delta^\lambda_\nu + \gamma^\mu[\delta^\lambda_\nu\partial_\mu + \Gamma^\lambda_{\mu\nu}])\psi^\nu] \\ &= 0 \ . \end{aligned} \tag{16}$$

Integrating by part and using

$$\overline{\Gamma}^\mu_{\lambda\mu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\sqrt{-g} \ ,$$

we get gravitational equations

$$\partial_\rho R_\lambda{}^{\mu\nu\rho} - \Gamma^\eta_{\rho\lambda}R_\eta{}^{\mu\nu\rho} + \Gamma^\mu_{\rho\eta}R_\lambda{}^{\eta\nu\rho} + \overline{\Gamma}^\rho_{\eta\rho}R_\lambda{}^{\mu\nu\eta} = \bar{\psi}_\lambda\gamma^\mu\psi^\nu \ , \tag{17}$$

and so

$$\partial_\rho R_\lambda{}^{\mu\nu\rho} = -\mathcal{J}^{\mu\nu}_\lambda \tag{18}$$

where $\mathcal{J}^{\mu\nu}_\lambda$ is the current

$$\mathcal{J}^{\mu\nu}_\lambda \equiv -\Gamma^\eta_{\rho\lambda}R_\eta{}^{\mu\nu\rho} + \Gamma^\mu_{\rho\eta}R_\lambda{}^{\eta\nu\rho} + \overline{\Gamma}^\rho_{\eta\rho}R_\lambda{}^{\mu\nu\eta} - \bar{\psi}_\lambda\gamma^\mu\psi^\nu \ , \tag{19}$$

which satisfies

$$\partial_\nu\mathcal{J}^{\mu\nu}_\lambda \equiv 0 \ . \tag{20}$$

4 The connection between the new gravitational equations and the Einstein equations and the weak field approximation

In the following we will discuss a special case when torsion becomes zero. Considering the property of the Riemann-Christoffel tensors and Antisymmetrizing the right of the gravitational equations, we get the gravitational equations without torsion:

$$D_\rho \overline{R}^\rho{}_{\lambda\mu\nu} = \frac{1}{2}[\bar{\psi}_\nu \gamma_\mu \psi_\lambda - \bar{\psi}_\mu \gamma_\nu \psi_\lambda] \tag{21}$$

for $D_\rho g_{\mu\nu} = 0$. In the vacuum without matter, the gravitational equations change into ($\overline{R}$ denotes no torsion)

$$D_\rho \overline{R}^\rho{}_{\lambda\mu\nu} = 0 \ . \tag{22}$$

We find on contraction of λ with ν in the Bianchi identities

$$D_\eta \overline{R}_{\lambda\mu\nu\kappa} + D_\kappa \overline{R}_{\lambda\mu\eta\nu} + D_\nu \overline{R}_{\lambda\mu\kappa\eta} = 0 \tag{23}$$

that

$$D_\eta \overline{R}_{\mu\kappa} - D_\kappa \overline{R}_{\mu\eta} + D_\nu \overline{R}^\nu{}_{\mu\kappa\eta} = 0 \ . \tag{24}$$

In empty space-time we get another form of the gravitational field equations from (22) (24)

$$D_\eta \overline{R}_{\mu\kappa} - D_\kappa \overline{R}_{\mu\eta} = 0 \ , \tag{25}$$

which were the equations derived by Yang in [3].

The Einstein field equations in empty space-time without matter are

$$\overline{R}_{\mu\nu} = 0 \ , \tag{26}$$

which satisfies

$$D_\eta \overline{R}_{\mu\kappa} - D_\kappa \overline{R}_{\mu\eta} = 0 \tag{27}$$

and also satisfies

$$D_\nu \overline{R}^\nu{}_{\mu\kappa\eta} = 0 \ , \tag{28}$$

from which we can see the Einstein equations are the special one of the quantum gravitational equations in the macroscopic scale.

In the weak stationary field, we can take linear approximation. Kept the linear terms in the left of the equations, the leading terms now are

$$\partial_\rho R^\rho{}_{t\mu t} = \frac{1}{2}[\bar{\psi}_t \gamma_\mu \psi_t - \bar{\psi}_\mu \gamma_t \psi_t] \tag{29}$$

let

$$\frac{1}{2}\bar{\psi}_\nu \gamma_\mu \psi_\lambda \equiv A_{\nu\lambda}\delta(\mu) \tag{30}$$

which demonstrates the singularity of the matter field and will lead to the creation of black hole naturally, the gravitational equations now change into

$$-\frac{1}{2}\partial_\mu \Box^2 g_{tt} \;=\; A_{tt}\delta(\mu)\ . \tag{31}$$

Integrating by dx^μ and letting $A_{tt} \equiv 4\pi G\rho_M$, we get

$$\nabla^2 g_{tt} \;=\; -8\pi G\rho_M \tag{32}$$

which is the Newton gravity.

5 Conclusion

From the above discussion, we see the energy-momentum tensors seem to be the source of gravity and the Newton's constant G is the effective coupling constant, which does not exist in nature, nor does the Planck scale. Gravity is not only a phenomenon of geometry, but also the result of geometry, which is created by the geometrical property of matter.

Acknowledgments:

One of the authors Cheng Jian-Feng acknowledges his special debt to Mr. Wu Yue-Liang for his generous help. He also owes thanks for help in finishing the paper to Huo W.J.and Yan Y.A..

References

[1] C.N. Yang and R.L. Mills, Phys. Rev. **96**, 191 (1954).

[2] S. Weinberg, Phys. Rev. **138**, B988 (1965).

[3] C.N. Yang, Phys. Rev. Lett. **33**, 445 (1974).

[4] Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D**12**, 3845 (1975).